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Lattice Boltzmann method for quantum field theory

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Abstract
It is shown that a (1 + 1)-dimensional lattice Boltzmann discretization of the Klein–Gordon and Dirac equations complies with equal-time-commutation-relations. As a result, the lattice Boltzmann discretization leads to a consistent lattice formulation of the quantum field theory in 1 + 1 dimensions. The present kinetic approach may offer a significant simplification over previous Hamiltonian formulations, because the quantum fields move along simple classical trajectories defined by the causal lightcones. This should facilitate both analytical manipulations and symbolic computer implementations of the method.

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1. Introduction

In the last decade, the lattice Boltzmann (LB) method has emerged as a powerful technique for the numerical simulation of complex flows [1].

Although the overwhelming majority of LB applications to date have been directed towards the study of classical fluid problems, over the years, a group of authors has also unravelled the potential of the LB representation for the solution of single- and many-body quantum-mechanical problems [2].

Quantum lattice Boltzmann (QLB) schemes are based on the idea of encoding the quantum wavefunction into a *complex* distribution, which is then evolved in close analogy with the classical Boltzmann distribution.

The advantage of such representation is that streaming (kinetic energy operator) proceeds along straight trajectories defined by classical lightcones, whereas interactions (potential energy) are completely local in space and time, as it is indeed appropriate for a relativistic field-theoretical framework. This leads to a very regular and structured spacetime flow of information, with a number of interesting properties, such as built-in causality and Lorentz invariance [3][1].

[1] Quantum lattice Boltzmann and quantum lattice gas algorithms come in two flavours, usually named type-I and type-II. The former addresses genuinely quantum problems, while the latter targets classical problems and can be reproduced by a classical LB scheme. In this sense, only type-I bears some relevance to the present work.

To date, QLB schemes have been confined to the framework of single-particle and many-particle quantum *mechanics*. In this paper, it is shown that the QLB formalism can be further extended to the case of quantum field theory, at least in 1 + 1-dimensions. This is achieved by promoting the complex distribution to the status of a complex-valued operator field, and showing that the discrete evolution of this operator field is compliant with equal-time-commutation-relations (ETCR).

As a result, QLB schemes may represent a new suitable candidate for the numerical/symbolic simulation of quantum field theory on a lattice.

2. Free Klein–Gordon equation

We begin by considering the Boltzmann formulation of the relativistic wave equation for a free massive particle in 1 + 1 dimensions [5, 6]:

$$\partial_t \psi^-(x,t) - c\partial_x \psi^-(x,t) = -\mathrm{i}\omega_c \psi^+ \tag{1}$$

$$\partial_t \psi^+(x,t) + c\partial_x \psi^+(x,t) = -\mathrm{i}\omega_c \psi^- \tag{2}$$

where $\omega_c = mc^2/\hbar$ is the Compton frequency of a material particle of mass m.

As it was shown in [5], a discrete unitary-scheme is obtained by integrating the streaming terms along the lightcones $\mathrm{d}x_j = c_j\,\mathrm{d}t$, where $j = \mp$ denote the left/right walker and $c_j = \mp c$, while the 'collision' term on the right-hand side is treated with a Cranck–Nicolson time marching (arithmetic average between times t and $t + \mathrm{d}t$).

The resulting discrete scheme can be cast in a compact transfer-matrix form (for details see [5]):

$$\psi_j(x + c_j\,\mathrm{d}t; t + \mathrm{d}t) = \sum_{k=\mp} T_{jk}\psi_k(x;t) \tag{3}$$

where x and t are discrete spacetime coordinates and T_{jk} is the discrete transfer matrix.

Note that the lightcone condition $\mathrm{d}x_j = c_j\,\mathrm{d}t$ guarantees that ψ_j lives on lattice sites $x_j = x + \mathrm{d}x_j$ at all discrete times $t_n = n\,\mathrm{d}t$.

The explicit expression of the transfer matrix elements is as follows:

$$\begin{aligned} T_{-,-} &= T_{+,+} = (1 - m^2/4)/(1 + m^2/4), \\ T_{-,+} &= T_{+,-} = -\mathrm{i}m/(1 + m^2/4). \end{aligned} \tag{4}$$

where we have set $m = \omega_c\,\mathrm{d}t$.

Let us now promote the scalars ψ_j to operator fields: the effect of the operator $\psi_j(x;t)$ is to generate a particle of speed c_j at spacetime location (x,t).

As a result, $\langle\psi_j(x,t)\rangle$ represents the probability of finding a particle of speed c_j at spacetime location (x,t), and the higher order moments $\mu_q(x,t) \equiv \langle\psi_j(x,t)^q\rangle - \langle\psi_j(x,t)\rangle^q$ provide the qth order fluctuations around this mean.

The full set of moments is tantamount to a complete knowledge of the quantum field.

According to the spirit of second quantization, in the above, the brackets stand for average over the *initial* configuration, so that the knowledge of the operator $\psi_j(x,t)$ in terms of $\psi_j(x, t = 0)$ provides full knowledge of the system observables at any time $t > 0$.

The basic requirement for the discrete operator equation (3) to define a consistent lattice quantum field theory is that the equal-time commutation relations (ETCR) be satisfied at all times. These read as follows [7–9]:

$$C^\star(x,y;t) \equiv [\psi^\star(x;t), \psi(y;t)] = \mathrm{i}\hbar\delta(x-y) \tag{5}$$

$$C(x, y; t) \equiv [\psi(x; t), \psi(y; t)] = 0 \tag{6}$$

$$C^{\star\star}, y; t) \equiv [\psi^\star(x; t), \psi^\star(y; t)] = 0 \tag{7}$$

In terms of the discrete velocity degrees of freedom, $\psi = \sum_j \psi_j$, the above relations turn into

$$C^\star(x, y; t) = \sum_{j,k=\mp} C^\star_{jk}(x, y; t), \tag{8}$$

where we have defined

$$C^\star_{jk}(x, y; t) = \psi^\star_j(x)\psi_k(y) - \psi_k(y)\psi^\star_j(x). \tag{9}$$

By assuming commutativity of different discrete velocity components, that is $[\psi^\star_j(x), \psi_k(y)] \propto \delta_{jk}$, relation (8) simplifies to

$$C^\star(x, y; t) = \sum_{j=\mp} C^\star{}_{jj}(x, y; t). \tag{10}$$

This is the ETCR to be conserved by the LB dynamics. Based on (3), one readily obtains (hereafter all indices run over $\mp$)

$$\psi^\star_j(x_j, t + \mathrm{d}t)\psi_j(y_j, t + \mathrm{d}t) = \sum_{k,l} T^\star_{jk} T_{jl} \psi^\star_k(x; t)\psi_l(y; t) \tag{11}$$

and

$$\psi_j(x_j; t + \mathrm{d}t)\psi^\star_j(y_j; t + \mathrm{d}t) = \sum_{k,l} T_{jl} T^\star_{jk} \psi_l(x, t)\psi^\star_k(y, t) \tag{12}$$

where we have used the shortands $x_j = x + c_j\,\mathrm{d}t$ and $y_j = y + c_j\,\mathrm{d}t$ to indicate the end points of the flight along the lightcone defined by c_j. Since the discrete-velocity commutators obey

$$[\psi^\star_j(x), \psi_k(y)] = \delta_{jk}[\psi^\star_j(x), \psi_j(y)],$$

we observe that, upon subtracting (12) from (11), the commutator at time $t + \mathrm{d}t$, reads as follows:

$$C^\star_{jj}(x_j, y_j; t + \mathrm{d}t) = \sum_k T^\star_{jk} T_{jk} C^\star_{kk}(x, y; t) \tag{13}$$

It is crucial to observe that $y_j - x_j = y - x$, so that the two commutators at times t and $t + \mathrm{d}t$ refer to the same spatial separation $r = y - x$. It is now a simple matter to check that the matrix elements (4) obey the sum rule $\sum_k |T_{jk}|^2 = 1$, so that

$$\sum_j C^\star_{jj}(x_j, y_j; t + \mathrm{d}t) = \sum_j C^\star_{jj}(x, y; t) \tag{14}$$

This proves the first ETCR in (5).

The proof of the second and third ETCR is even simpler, since the commutator at time $t + \mathrm{d}t$ is a linear combination of $[\psi_j(x, t), \psi_k(y, t)]$ (and the starred analogues).

Since at time $t = 0$ these are zero by definition, the commutator remains zero at all subsequent times.

It is useful to recast the lightcone ETCR relations (14) into a more conventional Eulerian form.

For this purpose, let us rewrite (14) in an explicit form as

$$C^\star_{-,-}(x - 1, y - 1; t + 1) + C^\star_{++}(x + 1, y + 1; t + 1) = C^\star_{-,-}(x, y; t) + C^\star_{++}(x, y; t) \tag{15}$$

where we have set $\mathrm{d}x = \mathrm{d}y = c\,\mathrm{d}t = 1$ for simplicity. Next, we observe that the commutation relations come in the homogeneus form:

$$C^{\star}(x, y; t) = \text{const.}W(|y - x|)$$

where $W(r)$ is a generic function of the distance $r = |y - x|$ (in actual terms, locality forces $W(r)$ to a Dirac's delta, but our argument applies for any generic function of r). As a result, (15) simplifies to

$$C^{\star}(r; t + 1) + C^{\star}(r; t + 1) = C^{\star}(r; t) + C^{\star}(r; t)$$

which is precisely the desired Eulerian form of the commutation relations. It is to be noted that the homogeneity of the commutation relations is essential for the LB scheme to be ETCR compliant.

And so is the mid-point rule in the time marching, exactly as in previous operator approaches to quantum field theory [8]. A few words of comment are in order.

The peculiar aspect of the present kinetic formulation is that each discrete field ψ_j only moves along its classical lightcone $\mathrm{d}x_j = c_j\,\mathrm{d}t$, notwithstanding the time-implicit formulation (mid-point rule in the time marching) of the problem.

This is because, thanks to the kinetic representation (fields do not mix in the free-streaming step), the mid-point rule does not generate a full-matrix problem, but a sequence of 2×2 linear problems at each lattice site, which are readily solved analytically to deliver the 'dressed' matrix coefficients T_{ij}.

This is expected to bring significant simplifications in the analytical treatment, as well as in symbolic computer implementations of the scheme.

3. The interacting Dirac equation

The relativistic Dirac equation for an interacting particle can be treated in a similar way, with the important caveat that the 'collision matrix' is no longer constant in spacetime because of the spacetime dependence on the electromagnetic field. For instance, in the case of electromagnetic interactions, the Dirac equation can be cast in the following complex LBE matrix form [4, 5]:

$$\partial_t \psi^-(x, t) - c\partial_x \psi^-(x, t) = \mathrm{i}g_- \psi^- - \mathrm{i}\omega_c \psi^+ \tag{16}$$

$$\partial_t \psi^+(x, t) + c\partial_x \psi^+(x, t) = \mathrm{i}g_+ \psi^+ - \mathrm{i}\omega_c \psi^- \tag{17}$$

where $g_\pm = q(\Phi \pm A)$, $\Phi(x, t)$ and $A(x, t)$ are the electrostatic and vector potentials respectively and q is the electric charge. In full analogy with the free massive case, one derives a LBE scheme with the following transfer matrix elements:

$$\begin{aligned}
T_{-,-} &= \left\{\left(1 - \mathrm{i}\frac{g_+}{2}\right)\left(1 + \mathrm{i}\frac{g_-}{2}\right) - m^2/4\right\}\Big/ M,\\
T_{+,+} &= \left\{\left(1 - \mathrm{i}\frac{g_-}{2}\right)\left(1 + \mathrm{i}\frac{g_+}{2}\right) - m^2/4\right\}\Big/ M,\\
T_{-,+} &= -T_{+,-} = -\mathrm{i}m/M,
\end{aligned} \tag{18}$$

where we have set $M = (1 + m^2/4 - g_- g_+/4 - \mathrm{i}(g_- + g_+)$. A simple calculation shows that the relations $|T_{-,-}|^2 + |T_{-,+}|^2 = 1$, $|T_{+,-}|^2 + |T_{+,+}|^2 = 1$, still hold, provided that the matrix elements are evaluated at the *same* spatial location $x = y$.[2]

[2] One must also secure that mixed commutators remain zero at all times, that is $C^*_{jk}(x, y; t + \mathrm{d}t) = 0$, $j \neq k$. This requires $\sum_l T^*_{jl} T_{kl} = 0$. Simple algebra shows that these relations are indeed fulfilled by the matrix elements (18).

From the formal point of view, the basic difference with respect to the free theory is that the matrix elements develop a spacetime dependence through the electro-magnetic field $A(x,t)$.

The same calculation as in the free case delivers

$$C^{\star}_{jj}(x_j, y_j; t+\mathrm{d}t) = \sum_{k=\pm} T^{\star}_{jk}(x; t) T_{jk}(y; t) C^{\star}_{kk}(x, y; t) \tag{19}$$

It is worth noting that, since we are dealing with a fermionic wavefunction, here the symbol C stands for *anti*-commutators.

Due to the linearity of the equations, the procedure previously illustrated for bosonic commutators carries over without modifications. In principle, the $(x - y)$ separation jeopardizes the ETCR's, because we cannot expect the relation $\sum_k T^{\star}_{jk}(x) T_{jk}(y) = 1$ to hold anymore, unless $x = y$.

Fortunately, *locality* of the commutation relations, i.e. the fact that $W(r) = \delta(r)$, saves the day because it forces x and y to be the same, so that ETCR's are again preserved. This indicates that, at variance with the free-particle case, the interacting theory *does* require locality, besides homogeneity, to make the LB scheme ETCR compliant.

4. Time marching

Time marching of the operator LB schemes (4), (18) proceeds by repeated iteration of the transfer matrix T, that is, symbolically

$$\psi(t + n\,\mathrm{d}t) = T^n \psi(t) \tag{20}$$

where all spatial and velocity indices have been relaxed for simplicity. The specific form of the matrix T^n can be computed explicitly by taking the nth power of the matrix T.

However, owing again to the lightcone structure of the kinetic representation, a simpler and more insightful algebraic formulation can be developed.

By applying the transfer matrix twice, we obtain (only the + component is reported for notational simplicity, the − being specularly symmetric in space):

$$\psi^+(x, t+2) = a^2\psi^+(x-2, t) + ab\psi^-(x-2, t) + ba\psi^-(x, t) + b^2\psi^+(x, t) \tag{21}$$

where we have set $\mathrm{d}x = \mathrm{d}t = 1$ and $a \equiv T_{+,+}$ and $b \equiv T_{+,-}$ to simplify the notation. A further iteration yields:

$$\begin{aligned}\psi^+(x, t+3) = {} & a^3\psi^+(x-3, t) + a^2b\psi^-(x-3, t) + 2ab^2\psi^+(x-1, t) \\ & + (a^2b + b^3)\psi^-(x-1, t) + ab^2\psi^+(x+1, t) + a^2b\psi^-(x+1, t)\end{aligned} \tag{22}$$

The hierarchical structure of the spacetime pattern corresponding to the LB scheme is visualized in figure 1.

From this figure, we see that $\psi^+(x, t+n)$ collects contributions from all sites $x_{nj} \equiv x - n + 2j$ at time t, with $j = 0, n-1$.

The contribution is given by the number of paths connecting (x_{nj}, t) to $(x, t+n)$ along the tree, each path contributing a weight $a^{n-k_\mp} b^{k_\mp}$, where $k_\mp$ is the number of 'up/down' or 'down/up' flips (i.e. collisions) undergone by the walker to keep its discrete speed $c_\pm$ aligned with the path. For instance, with reference to the free path {4210}, one has $k_+ = 0$ and $k_- = 1$, because the + component flies undisturbed all the way from 4 to 0, whereas the − component needs a flip at $(x-3, t)$ before it can fly aligned with path {4210}. As another example, it is readily checked that for path {5310}, one has $k_+ = 2$ and $k_- = 3$.

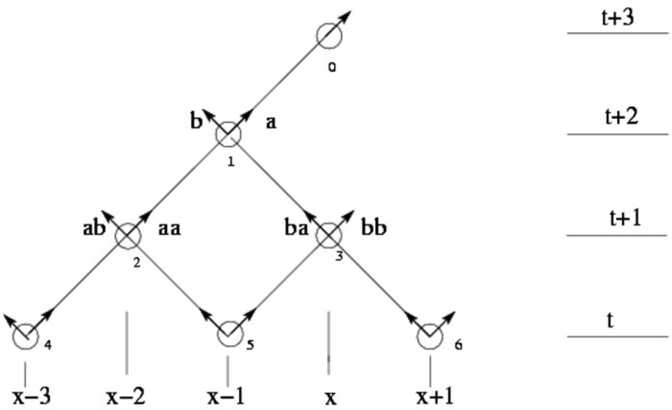

Figure 1. The hierarchical spacetime dependence for the evolution of the ψ^+ operator. Only three time levels are shown for simplicity.

Indeed, the '+' component flies freely in {53}, and needs two flips in {31} and {10}, whereas the − component needs a flip also in {53}, for a total of 3. With this type of book-keeping, the time-marching relation (20) can be recast in the form of a 'telescopic' (multi-step) path summation:

$$\psi^+(x, t+n) = \sum_{j=0}^{n-1} \sum_{p=0}^{P_{jn}} a^{n-k_+(p)} b^{k_+(p)} \psi^+(x_{nj}; t) + a^{n-k_-(p)} b^{k_-(p)} \psi^-(x_{nj}; t) \tag{23}$$

where the set of P_{jn} paths connecting $(x-n+2j, t)$ to $(x, t+n)$ is uniquely specified by the hierarchical tree structure depicted in figure 1. Of course, this telescopic representation can be applied over a number $n > 1$ of time slices only as long as the gauge field (Φ, A) does not feel any appreciable back reaction from ψ within a time scale $n\,\mathrm{d}t$.

For strongly interacting problems, a self-consistent time marching of *both* fields A and ψ is needed.

At this point, it is worth distinguishing between the case of background versus self-consistent interacting gauge fields.

The former does not require any equation for the gauge field, and can be handled with the procedure discussed above.

The latter, on the contrary, requires a major extension, whereby the evolution equation for operator wavefunction $A(x, t)$ should also be cast in kinetic form. For the case where $A(x, t)$ is a classical field, this task can be in a straightforward manner achieved by importing existing lattice schemes for classical wave-propagation [10].

The extensions to operator fields must secure compliance with the ETCR for bosonic fields.

Although a detailed calculation shall be left to future study, the fact that the gauge field obeys a wave equation (massless Klein–Gordon equation), with the simple addition of a local source term, $J = \psi^+\gamma\psi$, γ being the appropriate Dirac matrix, bodes well for the applicability of the present work to the self-consistent interacting case as well.

5. Nonlinear extensions

It is worth asking whether the present analysis can be extended to nonlinear problems in higher dimensions.

Both items represent major extensions, and consequently here we focus on the former one only. We start from the following generalization of (1):

$$\partial_+\psi_+ = -\mathrm{i}\omega_\mathrm{c}\mu(\psi_-) \tag{24}$$

$$\partial_-\psi_- = -\mathrm{i}\omega_\mathrm{c}\mu(\psi_+) \tag{25}$$

where $\partial_\mp = \partial_t \mp c\partial_x$ and $\mu()$ is a local nonlinear function of its argument. By acting with ∂_- and ∂_+ on the first and second equations, respectively, and summing up, we obtain

$$\partial^2(\psi_+ + \psi_-) = \mathrm{i}\omega_\mathrm{c}[\partial_-\mu(\psi_-) + \partial_+\mu(\psi_+)]$$

with $\partial^2 \equiv \partial_t^2 - c^2\partial_x^2$.

This reduces to the nonlinear Klein–Gordon equation for the symmetric combination $\Psi = \psi_+ + \psi_-$, under the constraint

$$\partial_-\mu(\psi_-) + \partial_+\mu(\psi_+) = \mathrm{const.}\mu(\psi_- + \psi_+). \tag{26}$$

This is a functional differential equation for the unknown function $\mu(\psi)$. It can be written as

$$\mu'(\psi_-)\mu(\psi_+) + \mu'(\psi_+)\mu(\psi_-) = \mathrm{const.}\mu(\psi_- + \psi_+) \tag{27}$$

where the prime denotes derivative over ψ.

Besides the expected linear solution $\mu(\psi) \propto \psi$, associated with the linear Klein–Gordon equation, the above equation delivers nonlinear solutions in the form $\mu(\psi) = \mathrm{e}^{k\psi}$, with k both being real and imaginary.

These generate sine-Gordon equations, whose corresponding LB form is

$$\psi_+(x, t+1) - \psi_+(x-1, t) = -\mathrm{i}m\frac{\mu_-(x-1, t) + \mu_-(x, t+1)}{2} \tag{28}$$

$$\psi_-(x, t+1) - \psi_-(x+1, t) = -\mathrm{i}m\frac{\mu_+(x+1, t) + \mu_+(x, t+1)}{2} \tag{29}$$

where the time step is made unit for simplicity ($\mathrm{d}x = c\,\mathrm{d}t = 1, m \equiv \omega_\mathrm{c}\,\mathrm{d}t$) and $\mu_\mp \equiv \mu(\psi_\mp)$.

Regrouping 'future' $(t+1)$ terms at the left-hand side, writing the equation for $\psi^*_\pm$ at position x and $\psi_\mp$ at position y, taking commutators at both left- and right-hand sides and summing them up, we finally obtain

$$\begin{aligned}
&[\psi_+^*(x, t+1), \psi_+(y, t+1)] + [\psi_-^*(x, t+1), \psi_-(y, t+1)] \\
&\quad = [\psi_+^*(x-1, t), \psi_+(y-1, t)] + [\psi_-^*(x+1, t), \psi_-(y+1, t)] \\
&\quad\quad - \frac{\mathrm{i}m}{2}(R_1(t+1) - R_1(t)) + \frac{m^2}{4}(R_2(t+1) - R_2(t))
\end{aligned} \tag{30}$$

where the first- and second-order residual commutators are defined as follows:

$$\begin{aligned}
R_1(t) = &-[\psi_+^*(x-1, t), \mu_-(y+1, t)] + [\mu_-^*(x+1, t), \psi_+(y-1, t)] \\
&- [\psi_-^*(x+1, t), \mu_+(y-1, t)] + [\mu_+^*(x-1, t), \psi_-(y+1, t)]
\end{aligned} \tag{31}$$

$$\begin{aligned}
R_1(t+1) = &-[\psi_+^*(x, t+1), \mu_-(y, t+1)] + [\mu_-^*(x, t+1), \psi_+(y, t+1)] \\
&- [\psi_-^*(x, t+1), \mu_+(y, t+1)] + [\mu_+^*(x, t+1), \psi_-(y, t+1)]
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
R_2(t) &= [\mu_-^*(x-1, t), \mu_-(y-1, t)] + [\mu_+^*(x+1, t), \mu_+(y+1, t)] \\
R_2(t+1) &= [\mu_-^*(x, t+1), \mu_-(y, t+1)] + [\mu_+^*(x, t+1), \mu_+(y, t+1)].
\end{aligned} \tag{33}$$

It is readily checked that in the linear case $\mu(\psi) \propto \psi$, $R_1(t) = 0$ by definition, whereas $R_1(t+1) = 0$ due to cross cancellations. On the other hand, $R_2(t)$ and $R_2(t+1)$ become proportional to the corresponding linear commutators, $[\psi_-^*(x+1,t), \psi_-(y+1,t)] + [\psi_+^*(x-1,t), \psi_+(y-1,t)]$ and $[\psi_+^*(x,t+1), \psi_+(y,t+1)] + [\psi_-^*(x,t+1), \psi_-(y,t+1)]$, so that they just contribute a proportionality constant. As a result, ETCR are indeed *exactly* satisfied.

In the general nonlinear case, however, ETCR are satisfied only to $O(m^2)$. This is readily seen by noting that $R_1(t) = 0$ still holds due to the operatorial identities $[f(a), b] = f'(a)[a, b]$, $[a, f(b)] = [a, b]f'(b))$.

However, to date, we have not succeeded in finding exact cancellations for $R_1(t+1)$.

Thus, by Taylor expanding, we obtain $R_1(t+1) - R_1(t) = O(m)$, so that the corresponding deviation from ETCR is $O(m^2)$.

Since R_2 is prefactored by m^2, the corresponding contribution $R_2(t+1) - R_2(t) = O(m^3)$.

This proves that the overall residual commutator is $O(m^2)$.

6. Computational perspectives

The practical use of the LB scheme presented in this work for the numerical/symbolic simulation of quantum field theories hinges upon the development of efficient strategies to compute the n-step propagator T^n. For a $(1+1)$-dimensional problem with N lattice sites and two-sided nearest-neighbour interactions, T^n is represented by a $N \times N$ matrix of bandwidth $2n+3$, whose elements are nth order polynomials of the matrix elements of T. As a result, a naive matrix–matrix product approach becomes quickly very demanding, and alternative methods to compute the action T^n on $\psi(0)$ without ever calculating T^n itself (matrix-free approach) must be devised.

Here again, a basic distinction between background and self-consistent gauge fields must be made. For the former, the one-sided nature of the LB operator T permits indeed to devise algebraic recipes, such as the path-summation expression (23). Another possibility, which remains to be explored for the future, is the diagonalization of T and subsequent use of the identity $T^n\psi = \lambda^n\psi$, ψ and λ being the eigenvectors/values of T. The case of self-consistent interactions, however, can only be handled step-by-step.

Given the Kth point equal-time correlator $\sigma_{i_1,i_2\ldots i_K}^{(p_1,p_2\ldots p_K)}(x_1, x_2 \ldots x_K; t) \equiv \left\langle \psi_{i_1}^{p_1}(x_1;t)\, \psi_{i_2}^{p_2}(x_2;t) \ldots \psi_{i_K}^{p_K}(x_K;t) \right\rangle$ at time t, the corresponding correlators at time $t+1$ can be computed in terms of T only. Even so, equal-time Kth order correlators are seen to require Pth order convolutions of T, with $P = \sum_{k=1}^{K} p_k$.

Since the operator fields ψ and A are 'frozen' at time t at this stage of the calculation, it is quite possible that algebraic path-summation strategies can be devised as well at each time step. However, to this point, this remains a conjecture which must be tested through detailed computations.

Summarizing, the present LB technique faces with the typical difficulties of the operator approach, with the potential advantage, though, of a simpler structure of the matrix representation of the evolution operator T.

Whether such an advantage is sufficient to make LB competitive against state-of-the art computational methods for quantum field theory, such as quantum and path-integral Monte Carlo, remains an open issue for future research.

7. Conclusions

Summarizing, we have shown that the lightcone lattice Boltzmann discretization of the 1+1-dimensional Dirac and Klein–Gordon equations is consistent with equal-time-commutation-relations . As a result, the LB scheme appears to be a suitable candidate for lattice simulations of (1 + 1)-dimensional quantum field theories. Since these latter are all but a mathematical nicety, with many applications in modern quantum physics [11], it is hoped that the present work may contribute a new entry to the simulation methods for lattice quantum field theories. Nonlinear extensions have also been presented in 1 + 1 dimensions for the case of the sine-Gordon equations. However, in such a case, ETCR are found to hold only at second order in the time step. Future work to extend the present procedure to general nonlinearities in $d + 1$ dimensions is certainly warranted.

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